

# Topological Structures in Yang Mills Magneto-Fluids

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Knotted configurations supported by a Yang Mills fluid-field system are suggested as a model for glueballs.

PACS numbers: 03.50.Kk, 11.10.Ef, 47.10.+g, 47.75.+f

## INTRODUCTION

The experimental discovery that quark-gluon plasmas (QGP) display features peculiar to strongly coupled fluids [1] has generated tremendous interest in building the non-Abelian equivalents of the standard, routinely used Abelian fluid models like magneto hydrodynamics (MHD) [2, 3, 4, 5, 6, 7, 8, 9, 13, 14, 15].

In one such recent model[2], the dynamics of a hot relativistic quark gluon fluid (with a non-Abelian charge) was described in terms of a generalized Yang Mills tensor born out of the unification of the gauge field and the flow-field tensor. The fact that it is possible to define a single non-Abelian unified field for the quark-gluon system strongly suggests that one should explore the system for topological structures endowed with properties such as linkages or knottedness of the fluid field lines that are preserved under ideal dynamics. If found, such a nonlinear stable soliton-like state could be identified with the wanted qcd object - the glueball [10, 11, 12]. The primary objective of this work is to construct and elucidate precisely such states as solutions to the fluid model [2]

In Abelian electrodynamics, the helicity of a vector field is the standard measure of the extent to which field lines coil around each other [16, 17]. The concept of helicity is trivially generalized to non-Abelian fields, and can serve as an index of topological complexity for the knotted solutions we are seeking. It is no wonder that the notion of helicity plays an important role in the study of plasma stability, and has been shown to have an intimate connection with knot theory.

The fluid-field model presented in [2] can be loosely considered as a non-Abelian generalization of the relativistic ( in directed as well as in thermal energy) fluid

description of electromagnetic plasmas [18]. The fluid equations are derived from a perfect fluid energy momentum tensor :  $T^{\mu\nu} = p\eta^{\mu\nu} + hU^\mu U^\nu$  with  $p$  as the pressure, and the enthalpy density  $h = mn_R f(T)$  where  $m$  and  $n_R$  are, respectively, the rest frame density and inertial mass of the particles comprising the fluid. The statistical attributes of the fluid are represented by the temperature dependent factor  $f(T) = f$ . Interestingly enough, for the important class of homentropic fluids,  $f$  appears in the equations of motion only as a multiplier to the fluid four velocity  $U^\mu$  (changing  $U^\mu$  to  $fU^\mu$ ); evidently the velocity displayed here carries a non-Abelian index ( $a$ ). Notice that the perfect fluid form for  $T^{\mu\nu}$  holds for both quark and gluon fluids, that is, for the gluon fields that have acquired a temperature dependent mass - the temperature dependence goes to define the appropriate  $f$  and the constant of proportionality may be viewed as the equivalent of the inertial mass in the expression for the enthalpy density. Naturally the massless gluon field is represented by the field tensor  $F^{\mu\nu}$

In Ref. [2], it was shown that the Lorentz force equation for a non-Abelian fluid takes the form

$$U^\alpha_a \left( \frac{m}{g} S^a_{\alpha\beta} + F^a_{\alpha\beta} \right) = 0. \quad (1)$$

In the equation of motion, the non-Abelian fluid tensor  $S^a_{\mu\nu}$

$$S^{\mu\nu}_a = \mathcal{D}^\mu(fU^\nu_a) - \mathcal{D}^\nu(fU^\mu_a) - imf^2[U^\mu_b, U^\nu_c], \quad (2)$$

where,  $\mathcal{D}_\mu$  is the generalized non-Abelian covariant derivative,  $\mathcal{D}_\mu = \partial_\mu - ig[A_\mu, ] - im[fU_\mu, ]$  appears on an equal footing with the standard field tensor  $F^a_{\alpha\beta}$ . We are, thus, lead naturally to a unified "minimally" coupled

potential for hot non-Abelian fluids

$$Q_a^\mu = A_a^\mu + \frac{m}{g} f U^\mu, \quad (3)$$

that generates its own unified fluid- field gauge tensor

$$M_a^{\mu\nu} = \partial_\mu Q_a^\nu - \partial_\nu Q_a^\mu + g c_a^{bc} Q_b^\mu Q_c^\nu. \quad (4)$$

It is pertinent to realize that  $S^a_{\mu\nu}$  contains the non-linear flow-field coupling through  $\mathcal{D}_\mu$  that depends on the Yang-Mills connection  $A_a^\mu$ .

We now have the machinery to explicitly construct topological fluid field solutions. Unlike the pure fluid or the pure Yang-Mills systems, the coupled system will sustain solutions ( similar to Magnetohydrodynamics) in which the fluid carries the Yang-Mills field with it, i.e the Yang Mills field is frozen in with the flow. Although  $SU(3)$  is the relevant group for the QGP, we solve here for illustration, the simpler problem for the symmetry group  $SU(2)$ .

We are interested in finding topologically nontrivial, and spatially localized solutions. The non-Abelian magneto fluid equation of motion(1) suggests  $\mathbf{M}_{\mu\nu} = \mathbf{0}$  to be a possible solution. In keeping with assumed localization of the solution, let us assume an interior and an exterior region. The exterior region extends out to infinity and applying traditional boundary conditions on fields at infinity, the proposed solution requires  $\mathbf{Q}_\mu \rightarrow \mathbf{0}$  at spatial infinity. Thus we can take  $\mathbf{Q}_\mu = \mathbf{0}$  (physical meaning will be dealt with later) in the entire exterior region.

The boundary between the interior and exterior regions is (without loss of generality), a three sphere and forms the overlap region for the interior and exterior solutions. Since  $\mathbf{Q}_\mu$  is a gauge connection, the interior solution  $\tilde{\mathbf{Q}}_\mu$  is related to the exterior solution  $\mathbf{Q}_\mu$  through a gauge transformation

$$\tilde{\mathbf{Q}}_\mu = \mathbf{\Omega} \mathbf{Q}_\mu \mathbf{\Omega}^\dagger - \frac{i}{g} \mathbf{\Omega} \partial_\mu \mathbf{\Omega}^\dagger. \quad (5)$$

It is not difficult to see that since  $\mathbf{M}_{\mu\nu} = g \mathbf{F}_{\mu\nu} + m \mathbf{S}_{\mu\nu}$  (being the curvature of the generalized connection  $\mathbf{Q}_\mu = \mathbf{A}_\mu + \frac{m}{g} f \mathbf{U}_\mu$ , while  $\mathbf{A}_\mu$  is the Yang-Mills connection), transforms covariantly. The generalized connection, by virtue of being a connection, transforms inhomogeneously and implies that  $f \mathbf{U}_\mu$ , the velocity vectors must transform covariantly. The inhomogeneous terms in the transformation of  $\mathbf{Q}_\mu$  are to be clubbed with the transformation of the Yang-Mills connection  $\mathbf{A}_\mu$ .

Thus, for the solution we are developing  $\mathbf{Q}_\mu = \mathbf{0}$  in the exterior, and  $\tilde{\mathbf{Q}}_\mu$  is pure gauge in the interior (both imply  $\mathbf{M}_{\mu\nu} = \mathbf{0}$ ) The overlap region being a three sphere  $\mathbb{S}^3$  then tells us that the group element  $\mathbf{\Omega}$ , belongs to the homotopy type given by the maps  $\mathbf{\Omega} : \mathbb{S}^3 \rightarrow SU(2)$  having chosen the gauge group to be  $SU(2)$ . The group manifold of  $SU(2)$  is isomorphic to the three sphere and we are left with the maps  $\mathbf{\Omega} : \mathbb{S}^3 \rightarrow \mathbb{S}^3$ . Such maps

are labelled by an integer, the "winding number" ( $n$ ) of the topological solution. Therefore our goal is to find an  $\mathbf{\Omega}$  such that its winding number is nonzero with the implication that the exterior solution  $\mathbf{Q}_\mu = \mathbf{0}$  cannot be extended into the interior.

For a pure gauge field, the winding (or, Pontryagin) number is simply given by

$$n = \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{Tr}[(\mathbf{\Omega} \partial_i \mathbf{\Omega}^\dagger)(\mathbf{\Omega} \partial_j \mathbf{\Omega}^\dagger)(\mathbf{\Omega} \partial_k \mathbf{\Omega}^\dagger)]. \quad (6)$$

Since the interior solution is simply given by

$$\tilde{\mathbf{Q}}_\mu = \frac{-i}{g} \mathbf{\Omega} \partial_\mu \mathbf{\Omega}^\dagger, \quad (7)$$

and  $\mathbf{\Omega} : \mathbb{S}^3 \rightarrow \mathbb{S}^3$ , we automatically find that the solution satisfies  $\tilde{\mathbf{Q}}_0 = 0$ . While in the exterior, we have required  $\mathbf{Q}_\mu = \mathbf{0}$ . A discussion of these conditions will be given below.

To construct an explicit "pure gauge" solution whose winding number is nonzero, we will borrow from the study of instantons [19] in pure Yang-Mills theories. Taking

$$\mathbf{\Omega}(x) = \frac{|\vec{x}|^2 - 1}{1 + |\vec{x}|^2} + \frac{2i\vec{\sigma} \cdot \vec{x}}{1 + |\vec{x}|^2} \quad (8)$$

where  $\vec{\sigma}$  are the Pauli matrices, it is easy to see that the  $SU(2)$  gauge components of  $\mathbf{Q}_\mu$  are given by

$$\mathbf{Q}^1 = \frac{-4}{g(\vec{x}^2 + 1)^2} ((\frac{1}{2}(1 - |\vec{x}|^2) + x^2)\hat{x} + (xy + z)\hat{y} + (xz - y)\hat{z}) \quad (9)$$

$$\mathbf{Q}^2 = \frac{-4}{g(1 + \vec{x}^2)^2} ((xy - z)\hat{x} + (\frac{1}{2}(1 - |\vec{x}|^2) + y^2)\hat{y} + (yz + x)\hat{z}) \quad (10)$$

$$\mathbf{Q}^3 = \frac{-4}{g(1 + \vec{x}^2)^2} ((xz + y)\hat{x} + (yz - x)\hat{y} + (\frac{1}{2}(1 - |\vec{x}|^2) + z^2)\hat{z}) \quad (11)$$

As we have seen above, the time component of  $\mathbf{Q}$ ,  $Q^0 = 0$ . It is easy to see that  $\int_{\mathbb{S}^3} \mathbf{Q}^1 \wedge \mathbf{Q}^2 \wedge \mathbf{Q}^3 = \int \frac{64}{g^3(1 + \vec{x}^2)^3} dx \wedge dy \wedge dz = \frac{2\pi^2}{g^3}$ . From equation 7, the fact that the  $SU(2)$  one form is  $\vec{Q} = \sigma_i Q^i$  and the properties of the product of three  $\sigma$  matrices, we can see that  $\int_{\mathbb{S}^3} \mathbf{Q}^1 \wedge \mathbf{Q}^2 \wedge \mathbf{Q}^3 = \frac{2\pi^2}{g^3} n$ , where  $n$  is given in eqn.6. Thus the winding number of this fluid field knot is  $n = 1$ . We illustrate the nature of this solution by plotting in figs.1,2 and 3 (with a composite plot in fig.4), the surfaces (in toroidal coordinates) on which the  $Q^i$  lie.

For each of the gauge fields  $\mathbf{Q}^i$  we can find three variables  $\alpha^i, \beta^i$  and  $\psi^i$  such that  $\mathbf{Q}^i = \alpha^i \vec{\nabla} \beta^i + \vec{\nabla} \psi^i$  (no

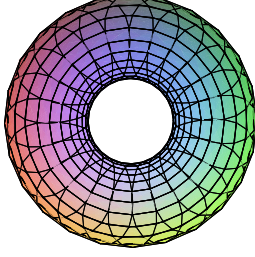


FIG. 1: shows the surface on which  $Q_1$  lies.

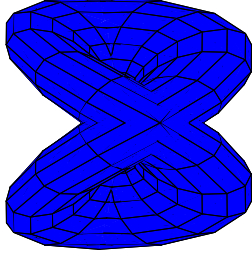


FIG. 2: shows the surface on which  $Q_2$  lies.

summation in  $i$ ). Explicit expressions are:

$$\begin{aligned}
 \alpha^1 &= \tan^{-1}\left(\frac{z}{y}\right) + \tan^{-1}\left(\frac{2x}{(1-\bar{x}^2)}\right) \\
 \alpha^2 &= \tan^{-1}\left(\frac{x}{z}\right) + \tan^{-1}\left(\frac{2y}{(1-\bar{x}^2)}\right) \\
 \alpha^3 &= \tan^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{2z}{(1-\bar{x}^2)}\right) \\
 \beta_1 &= \frac{z^2 + y^2}{g(1+\bar{x}^2)}; \beta_2 = \frac{x^2 + z^2}{g(1+\bar{x}^2)} \\
 \beta_3 &= \frac{y^2 + x^2}{g(1+\bar{x}^2)} \\
 \psi^i &= \frac{1}{4g} \tan^{-1}\left(\frac{\bar{x}^2 - 1}{2x_i}\right)
 \end{aligned} \tag{12}$$

The representation is clearly Clebsch like with the caveat that  $\psi^i$  is multiple valued, and therefore contributes a

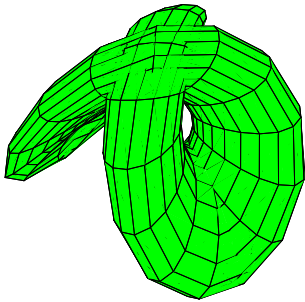


FIG. 3: shows the surface on which  $Q_3$  lies.

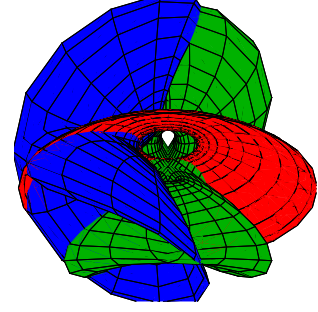


FIG. 4: shows the surface on which all the three vector fields,  $Q_1$ ,  $Q_2$ , and  $Q_3$  lie.

non vanishing contribution to the helicity when integrated over a closed contour. This decomposition allows us to separate the contributions of the field and the fluid. Such structures for the pure gauge field ( $A_i$ ) have been used in magnetohydrodynamics to find third order linkages between three magnetic fields in magnetic recombination and geophysical processes [20, 21, 22]. In these works, each component of the  $SU(2)$  field is considered to be a  $U(1)$  Abelian magnetic field.

Once the solution with winding number  $n = 1$  has been found, the solutions with higher order winding numbers  $n$  can be obtained by applying the gauge transformation  $\Omega(x)^{n-1}$  to the  $n=1$  solution [19]. Thus a whole spectrum of fluid field knots with integral winding numbers can be produced. It should be emphasized that these knots consist of both the velocity of the fluid and the gauge potential of the Yang-Mills field. In the region exterior to the knot, the gauge potentials have to satisfy appropriate constraints.

The solutions we have constructed, therefore, have the intended character; they are localized as well as topologically nontrivial. The exterior solution ( $Q^a_\mu = \frac{m}{g} f U^a_\mu + A^a_\mu = 0$ , implying  $j^a_\mu \propto A^a_\mu$ ) is the non-Abelian analog of the London equation, and displays, what might be viewed as an “inverse” Meissner effect; the “magnetic” flux is pushed out of the exterior region into the interior region of the knots, which can be regarded as chromomagnetic knotted flux tubes.

Because the unified connection  $\mathbf{Q}$  combines the fluid and the Yang Mills fields, the vanishing of its time component is not an empty condition; it implies that the time component of the gauge potential is proportional to the time component of the species velocity. This again can be viewed as a generalized Coulomb gauge condition providing for the staticity of the solution. Since the Pontryagin number labels these topological solutions, it is suggestive to consider it as a quantum number for such solutions.

There has been recent speculation that glueballs in a Yang-Mills theory may just be such topological, knotted solutions with their energies providing an analog of the energy levels of Bohr’s atom [11, 23, 24]. A detailed

estimate of the energies of our topological solutions is yet to be carried out, but there does exist a radial length scale in the theory given by  $\frac{m}{g}$ , which should lead to a non zero minimum energy of these knots.

Knotted solutions, developed in this work, should exist not only in the QGP, but also in quark stars and the early universe. A detailed investigations of the physical properties of these fluid field knots figures to be important for all strongly coupled quark- gluon matter.

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